3.6 The adjoint representation

We start by introducing some general terminology.

Definition 3.99

1) A representation of a Lie group G in a real or complex vector space V is a smooth homomorphism

$$\Pi : G \longrightarrow GL(V)$$

2) A representation of a Lie algebra $\mathfrak{g}$ in a real or complex vector space V is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

Remark 3.100

We have seen that a representation $\Pi : G \longrightarrow GL(V)$ induces a representation $D\Pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of the Lie algebra.

Recall that the Lie group exponential in.

$GL(V)$ coincides with the matrix exponential.

Then: $\pi(\exp_G(X)) = \mathrm{Exp}(D\pi(X))$
$\forall X \in \mathfrak{g}$

by Proposition 3.59.

Let now $W \subset V$ be a vector subspace and fix $v \in V$. Let

$$\mathrm{Stab}(W) := \{ g \in G : \pi(g).W \subset W \}$$

$$\mathrm{Stab}(v) := \{ g \in G : \pi(g)v = v \}.$$

Note that $\mathrm{Stab}(W)$ and $\mathrm{Stab}(v)$ are closed subgroups of G and the following holds:

Proposition 3.101

$$\mathrm{Lie}(\mathrm{Stab}(W)) = \{ x \in \mathfrak{g} : D\pi(x) W \subset W \}$$

$$\mathrm{Lie}(\mathrm{Stab}(v)) = \{ x \in \mathfrak{g} : D\pi(x) v = 0 \}.$$

Before discussing the proof note that for $g \in G$ $\pi(g) W \subset W$ iff $\pi(g) W = W$.

Proof of Proposition 3.101

By Corollary 3.96

$$\mathrm{Lie}(\mathrm{Stab}(W)) = \{ X \in \mathfrak{g} : \pi(\exp_G tX) W \subset W \quad \forall t \in \mathbb{R} \}$$
$$= \{ X \in \mathfrak{g} : (\mathrm{Exp}\, t D\pi(X)) W \subset W \ \forall t \in \mathbb{R} \}$$

Then we note (Exercise) that for $A \in \mathfrak{gl}(V)$

$$\mathrm{Exp}\, tA (W) \subset W \quad \forall t \in \mathbb{R}$$
$$\text{iff} \qquad A(W) \subset W.$$

The conclusion follows.

In order to prove 2) we argue in a similar way.

By Corollary 3.96
$$\mathrm{Lie}(\mathrm{Stab}(v)) = \{ X \in \mathfrak{g} : (\mathrm{Exp}\, t D\pi(X)) v = v \quad \forall t \in \mathbb{R} \}.$$

Then a similar Exercise shows that

$(\operatorname{Exp} tA) v = v \quad \forall\, t \in \mathbb{R}$

iff $\quad A(v) = 0 \,.$ $\square$

We turn to the study of the adjoint representation.

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For $g \in G$ we denote as usual $\operatorname{int}(g) : G \longrightarrow G$

$$x \longmapsto g x g^{-1}$$

Note that $\operatorname{int}(g)$ is a smooth automorphism of G and we denote by

$$\operatorname{Ad}(g) := D_e \operatorname{int}(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

its derivative.

We clearly have $\operatorname{Ad}(e) = \operatorname{Id}_{\mathfrak{g}}$ and the chain rule gives $\operatorname{Ad}(g_1 g_2) = \operatorname{Ad}(g_1) \operatorname{Ad}(g_2)$ $\forall\, g_1, g_2 \in G$.

$\boxed{\text{Exercise } 3.102}$

$$G \longrightarrow GL(\mathfrak{g})$$
$$g \longmapsto \operatorname{Ad}(g).$$

is a smooth homomorphism.

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From Proposition 3.59 we deduce the so called fundamental relation:

$\forall g \in G \quad \forall t \in \mathbb{R} \quad \forall X \in \mathfrak{g}$

$$(*) \quad g \exp_G(tX) g^{-1} = \exp_G(t \operatorname{Ad}(g) X).$$

With the help of $(*)$ we can compute Ad for $G = GL(n,\mathbb{R})$. For $g \in GL(n,\mathbb{R})$, $X \in \mathfrak{gl}(n,\mathbb{R})$ and $t \in \mathbb{R}$

$$\sum_{k=0}^{\infty} \frac{t^k (\operatorname{Ad}(g) X)^k}{k!} = g \left(\sum_{k=0}^{\infty} \frac{t^k X^k}{k!} \right) g^{-1}$$

$$= \sum_{k=0}^{\infty} \frac{t^k (g X g^{-1})^k}{k!}.$$

Comparing coefficients of the two power series this gives

$$\operatorname{Ad}(g) X = g X g^{-1}.$$

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In analogy with the G-action by conjugation on G a Lie algebra acts on itself by the Lie bracket.

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}),$$
$$X \longmapsto [X, \cdot].$$

This is the so called adjoint representation of $\mathfrak{g}$ and the fact that it is a Lie algebra homomorphism is equivalent to the Jacobi identity (Exercise).

Theorem 3.103

Let G be a Lie group, $\mathrm{Ad}: G \longrightarrow GL(\mathfrak{g})$ its adjoint representation and $\mathrm{ad}: \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ the adjoint representation of its Lie algebra.

Then:
$$D_e \mathrm{Ad}(X) = \mathrm{ad}(X) \quad \forall X \in \mathfrak{g}.$$

Proof

We start by noting that in general if $\rho: G \longrightarrow GL(V)$ is a representation then:

$$D\varphi(X) = \frac{d}{dt}\Big|_{t=0} \varphi(\exp(tX)) \quad \forall X \in \mathfrak{g}.$$

This follows from Proposition 3.59.

Thus in our setting.

$$D_e Ad(X) = \frac{d}{dt}\Big|_{t=0} Ad(\exp(tX))$$

and therefore. $\forall\, Y \in \mathfrak{g}$

$$D_e Ad(X)(Y) = \left(\frac{d}{dt}\Big|_{t=0} Ad(\exp(tX))\right)(Y)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(Ad(\exp(tX))\right)(Y).$$

$$= \frac{d}{dt}\Big|_{t=0} D_e c_{\exp tX}(Y).$$

$$= \frac{d}{dt}\Big|_{t=0} DR_{\exp(-tX)} DL_{\exp tX}(Y).$$

$$= \frac{d}{dt}\Big|_{t=0} DR_{\exp(-tX)} Y^L_{\exp tX}.$$

Thus if we denote by $\Phi^{X^L}_t$ the flow associated to the left invariant.

extension X^L of X we have.

$$D_e\, Ad(X)_\cdot(Y) = \frac{d}{dt}\Big|_{t=0} DR_{\exp(-tX)}\, Y^L_{\exp tX}$$

$$= \frac{d}{dt}\Big|_{t=0} D_{\Phi^{X^L}_t(e)} \Phi^X_{-t}\left(Y^L_{\Phi^X_t(e)}\right)$$

$$= L_{X^L} Y^L(e) \quad \leftarrow \text{Def } 3.49.$$

$$= [X, Y].$$

Theorem 3.50

$\square$

$\boxed{\text{Corollary } 3.104}$

1) Let $N < G$ be a closed subgroup with the Lie algebra $\mathfrak{n} < \mathfrak{g}$. If N is normal then $\mathfrak{n}$ is an ideal of $\mathfrak{g}$.

2) If G and N are connected and $\mathfrak{n} < \mathfrak{g}$ is an ideal then $N \trianglelefteq G$.

Proof

The proof of 1) is left as an Exercise.

In order to prove 2) note that the condition that $\mathfrak{n}$ is an ideal can

be rephrased by saying that $ad(X)$ leaves $\mathfrak{n}$ invariant $\forall X \in \mathfrak{g}$.
Then so does $\operatorname{Exp}(ad(X))$ and hence $Ad(\exp(X)) = \operatorname{Exp}(ad(X))$ also leaves $\mathfrak{n}$ invariant.

Thus $\forall X \in \mathfrak{g}$ $\forall Y \in \mathfrak{n}$.

$$Ad(\exp(X))(Y) \in \mathfrak{n}$$

which implies that

$$\exp[Ad(\exp X)(Y)] \in N$$

However by the fundamental relation we have.

$$\exp[Ad(\exp X)(Y)] = \operatorname{int}_{\exp X}(\exp Y).$$

Thus the subgroup of G generated by $\exp \mathfrak{g}$ leaves the subgroup of N generated by $\exp \mathfrak{n}$ invariant. By connectedness $N \triangleleft G$. $\square$

We conclude with two applications.

Theorem 3.105

Let G be a connected Lie group. Then the center of G is the kernel of the adjoint representation.

Proof

Let $h \in Z(G)$ and $X \in \mathfrak{g}$. Then

$$(*) \quad \exp tX = h \exp tX \, h^{-1} = \exp t \, \mathrm{Ad}_h(X)$$
$\forall \, t \in \mathbb{R}$.

Hence $X = \mathrm{Ad}_h(X) \quad \forall \, X \in \mathfrak{g}$.

Therefore $h \in \mathrm{Ker}(\mathrm{Ad})$.

Conversely assume that $h \in \mathrm{Ker}\,\mathrm{Ad}$. Then $(*)$ holds again and we infer that h commutes with all the elements in a neigh of $e \in G$. Since G is connected $h \in Z(G)$. $\square$

Corollary 3.106

Let G be a connected Lie group. Then $Z(G)$ is a closed subgroup of G with

Lie algebra. the center of $\mathfrak{g}$.

Proof

The statement follows from Theorem 3.105 and Corollary 3.97. $\square$